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Polarisation of synchrotron radiation by a Dirac particle

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Abstract. We consider the polarisation of synchrotron radiation by a Dirac particle. By making use of the Stokes parameters, we are able to pick out the linearly polarised components from the circularly polarised component. Particular attention is paid to the case of large energy. Since we are dealing with a Dirac particle, interest is also shown in the role of the spin in the polarisation phenomenon.

1. Introduction

Since Schott obtained his formula, considerable interest has been aroused in the theoretical examination of the problem of radiation by a relativistic particle circulating in a uniform magnetic field. This interest seemed particularly strong during the 1940's as a result of the construction of particle accelerators to ever increasing energies. But notwithstanding the many and varied volumes of literature on the subject, interest in synchrotron radiation does not seem to abate. Since the 1970's rapid expansions have been taking place in synchrotron radiation facilities with an intensity that seems even to surpass that of the 1940's. This current upsurge of interest is due to the recognition that synchrotron radiation can be used as a source for the study of photo-nuclear reactions and indeed in the physics of high energy generally. It has been suggested that unique experimental conditions in photo-nuclear research would result from it because of certain properties of synchrotron radiation. These properties include the very high intensity, a high degree of polarisation and the extremely good collimation of the radiation.

It may be noted in this regard that most of the existing theoretical works on the subject concentrate on the angular and spectral distribution of the radiation. There is no corresponding interest shown in a study of the polarisation properties of the radiation particularly by the quantum treatment. Information on this aspect is principally experimental. Such a study is of some interest in view of a possible relation between the spin state of a charged particle and the polarisation state of its radiation. This is what we intend to examine in this article.

Since the significance of Dirac's equation lies in the fact that it treats the spin as an intrinsic property of the electron, it provides a natural starting point for our examination. Any solution should unfold these properties naturally. Thus, although numerous solutions of this equation exist, and indeed for the case of a particle moving in a magnetic field, we intend to obtain a fresh solution here for the sake of consistency and continuity. Such an approach will enable us to emphasise those points which we have to use in our subsequent analysis. Further, for the purpose of studying the polarisation properties of the radiation, we shall apply the general methods of describing the polarisation properties of a bundle of light as demonstrated by Stokes (see e.g. McMaster 1954). Finally, in view of current trends, we shall devote our analysis to the case of large energies and possibly also high magnetic fields.

2. Solution of Dirac's equation

If we represent the wavefunction in the form of a column matrix

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} e^{-iEt/\hbar}$$

then Dirac's equation for a stationary particle gives the following system of equations:

$$(E-\mu c^2)\psi_1 - cP_-\psi_4 - cP_z\psi_3 = 0, \qquad (E-\mu c^2)\psi_2 - cP_+\psi_3 - cP_z\psi_4 = 0,$$

$$(E+\mu c^2)\psi_3 - cP_-\psi_2 - cP_z\psi_1 = 0, \qquad (E+\mu c^2)\psi_4 - cP_+\psi_1 - cP_z\psi_2 = 0.$$

Here μ is the rest mass of the particle, and $P_\pm = P_x \pm iP_y$ where $\boldsymbol{P} = \boldsymbol{P} - (e/c)\boldsymbol{A}$ is the momentum of the particle in the presence of the applied magnetic field.

Eliminating ψ_1 and ψ_3 from the first of the above equations we obtain for ψ_4

$$[c^2(P_+P_- + P_z^2) - (E^2 - \mu^2c^4)]\psi_4 = 0. \tag{1}$$

It can be shown that the component ψ_2 satisfies this same equation. We may therefore seek solutions in which the components ψ_2, ψ_4 differ only by a constant factor of proportionality.

For the components ψ_1, ψ_3 we similarly obtain

$$[c^2(P_-P_+ + P_z^2) - (E^2 - \mu^2c^4)]\psi_{1,3} = 0, \tag{2}$$

in view of which we may impose the further requirement that ψ_1 and ψ_3 also differ only by a constant factor. We shall see later (equation (9)) that a complete set of helicity eigensolutions can be obtained under this double restriction.

If we direct the magnetic field along the z axis then

$$A_x = -\tfrac{1}{2}y\mathscr{H}, \qquad A_y = \tfrac{1}{2}x\mathscr{H}, \qquad A_z = 0,$$

where $\mathscr{H}$ is the applied magnetic field. In obtaining (1) and (2), we had assumed that the magnetic field is directed along the z axis. We shall consider the case of an electron, i.e. $e = -e_0$, such that $e_0 > 0$ and $e^2 > 0$.

Equations (1) and (2) become

$$[\boldsymbol{\Delta} + k^2 - (e_0\mathscr{H}/\hbar^2c)(L_z - \hbar) - (e^2\mathscr{H}^2/4\hbar^2c^2)(x^2+y^2)]\psi_{2,4} = 0,$$

$$[\boldsymbol{\Delta} + k^2 - (e_0\mathscr{H}/\hbar^2c)(L_z + \hbar) - (e^2\mathscr{H}^2/4\hbar^2c^2)(x^2+y^2]\psi_{1,3} = 0.$$

Here $k^2 = K^2 - k_0^2$ and $E = \hbar cK$ is the energy of the particle, $\hbar k_0 = \mu c$ and $\boldsymbol{\Delta}$ is the Laplacian operator. Further, $L_z = xp_y - yp_x$ is the component of angular momentum in the direction of the z axis. In cylindrical coordinates, these equations take the form

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} + k^2 - 2\alpha\left(\frac{1}{i}\frac{\partial}{\partial\varphi} - 1\right) - \alpha^2r^2\right]\psi_{2,4} = 0,$$

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}+k^2-2\alpha\left(\frac{1}{\mathrm{i}}\frac{\partial}{\partial\varphi}+1\right)-\alpha^2r^2\right]\psi_{1,3}=0$$

where $\alpha = e_0\mathscr{H}/2\hbar c$.

In subsequent calculations, we shall ignore the z component.

We seek for solutions in the following forms:

for (1) we write $\psi_{2,4}=(\mathrm{e}^{\mathrm{i}(m+1)\varphi}/\sqrt{2\pi})R_{2,4}^{(r)}$;

for (2) $\psi_{1,3}=(\mathrm{e}^{\mathrm{i}m\varphi}/\sqrt{2\pi})R_{1,3}^{(r)}$.

we then obtain expressions for the radial parts thus:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{(m+1)^2}{r^2}+k^2-2\alpha m-\alpha^2r^2\right)R_{2,4}=0, \tag{3}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{m^2}{r^2}+k^2-2\alpha(m+1)-\alpha^2r^2\right)R_{1,3}=0. \tag{4}$$

Next we make the following substitutions in succession.

$R=\chi/r^{1/2}$, $\chi_{2,4}=r^{m+3/2}\exp(-\frac{1}{2}\alpha r^2)U_{2,4}$, $\chi_{1,3}=r^{m+1/2}\exp(-\frac{1}{2}\alpha r^2)U_{1,3}$,

where U is a function of r only. We then obtain the following equations for the U:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+2\left(\frac{m+\frac{3}{2}}{r}-\alpha r\right)\frac{\mathrm{d}}{\mathrm{d}r}+k^2-4\alpha(m+1)\right]U_{2,4}=0,$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+2\left(\frac{m+\frac{1}{2}}{r}-\alpha r\right)\frac{\mathrm{d}}{\mathrm{d}r}+k^2-4\alpha(m+1)\right]U_{1,3}=0.$$

Finally the substitution $p=\alpha r^2$ takes us to the form

$$[p\mathrm{d}^2/\mathrm{d}p^2+(m+2-p)\mathrm{d}/\mathrm{d}p+\tilde{k}^2-(m+1)]U_{2,4}=0,$$

$$[p\mathrm{d}^2/\mathrm{d}p^2+(m+1-p)\mathrm{d}/\mathrm{d}p+\tilde{k}-(m+1)]U_{1,3}=0.$$

These two equations can be combined into one:

$$[p\mathrm{d}^2/\mathrm{d}p^2+(m_j+\tfrac{1}{2}+1-p)\mathrm{d}/\mathrm{d}p+\tilde{k}^2-(m+1)]U_\varepsilon=0 \tag{5}$$

where for $\varepsilon=2,4$; $m_j=m+\frac{1}{2}$ and when $\varepsilon=1,3$; $m_j=m-\frac{1}{2}$.

Further

$$\tilde{k}^2=k^2/4\alpha=(K^2-k_0^2)/4\alpha.$$

Equation (5), as is well known, is satisfied by the generalised Laguerre polynomial, since m is an integer. Thus we have

$$U_\varepsilon=L_n^{m_j+1/2}(\alpha r^2)$$

where $n=0,1,2,3\ldots$ is the radial quantum number. The requirement that the Laguerre function be a polynomial leads to the energy levels. Thus

$$E_{n,m}=c[\mu c^2+2\mu\hbar\tilde{\omega}(n+m+1)]^{1/2}, \qquad \tilde{\omega}=e_0\mathscr{H}/\mu c. \tag{6}$$

Since the Dirac equation also describes the spin properties of the particle, the function ψ depends not only on the space coordinates, but also on the 'spin coordinates'. The normalisation must therefore make provision for this. Thus for the time-independent part of the function we must have

$$\psi_\varepsilon = \{\exp[\mathrm{i}(m_j + \tfrac{1}{2})\varphi]/\sqrt{2\pi}\} b_\varepsilon R_\varepsilon(r)$$

where $\boldsymbol{b}$ represents the spin amplitude and is independent of the spatial coordinates. It satisfies the condition

$$\boldsymbol{b}^+\boldsymbol{b} = b_1^* b_1 + b_2^* b_2 + b_3^* b_3 + b_4^* b_4 = 1. \tag{7}$$

This condition is however not enough to determine the four unknown quantities. We therefore make use of the fact that ψ is also an eigenfunction of the operator of the projection of the spin on the particle momentum, namely

$$(\boldsymbol{\sigma} \cdot \boldsymbol{P})\psi = \hbar k s \psi.$$

Since this operator commutes with the Hamiltonian of Dirac's equation, with the help of Dirac's equation, the latter equation can be transformed to the form

$$(p_1 E/c + \mathrm{i} p_2 \mu c)\psi = \hbar k s \psi. \tag{8}$$

Equation (8) gives us a system of four linear equations in the b's, but only two of them are linearly independent. By substituting the solutions into the original equations we are able to get the following values for the b's:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} s(1+k_0/K)^{1/2} \\ \mathrm{i}s(1+k_0/K)^{1/2} \\ (1-k_0/K)^{1/2} \\ \mathrm{i}(1-k_0/K)^{1/2} \end{pmatrix}.$$

From the solutions of these equations we also find that $s = \pm 1$ and $K^2 = k^2 + k_0^2$ where $\hbar k$ is the momentum of the particle. Thus for the wavefunction we obtain

$$\psi(r, \varphi, t) = \frac{\exp[-cKt + \mathrm{i}(m_j + \frac{1}{2})\varphi]}{\sqrt{2\pi}} \begin{pmatrix} b_1 I_{n,m}(z) \\ b_2 I_{n,m+1}(z) \\ b_3 I_{n,m}(z) \\ b_4 I_{n,m+1}(z) \end{pmatrix} \tag{9}$$

where $I_{n,m}(z) = c_n z^{|m/2|} \exp(-\frac{1}{2}z) L_n^m(z)$ and $z = \alpha r^2$. The c_n are normalising constants.

3. Study of the polarisation

For the purpose of studying the polarisation properties of the radiation, we need to evaluate the instantaneous power of radiation. According to Sokolov *et al* (1968) this is given by

$$\mathrm{d}W = (e^2 c/2\pi)\, \mathrm{d}^3\boldsymbol{\varkappa}\, \delta(K_{n,mj} - K_{n',m'_j} - \varkappa) \oint. \tag{10}$$

where

$$\oint = (\boldsymbol{\alpha}_0^+ \boldsymbol{a})(\boldsymbol{\alpha}_0 \boldsymbol{a}^+)$$

with

$$\alpha^0 = \int \psi_j \, e^{-i\varkappa \cdot r} \alpha \psi_i \, d^3 r$$

being the matrix elements of the transition probability from the state ψ_i to the state ψ_j. Note that $\psi_i = \psi_i(r, \varphi)$.

Further, $\boldsymbol{a}$ is the amplitude of the wavefunction of the photon.

We split $\boldsymbol{a}$ into two orthogonal components, namely

$$\boldsymbol{a} = a_f \boldsymbol{e}_f + a_g \boldsymbol{e}_g$$

where

$$\boldsymbol{e}_f = \cos x \boldsymbol{\beta}_2 + e^{i\gamma} \sin x \boldsymbol{\beta}, \qquad \boldsymbol{e}_g = \sin x \boldsymbol{\beta}_2 - e^{i\gamma} \cos x \boldsymbol{\beta}_3. \tag{11}$$

We note that

$$\boldsymbol{e}_f^+ \cdot \boldsymbol{e}_g = 0.$$

Further, a_f and a_g are the operator parts of $\boldsymbol{a}$ and satisfy the commutation relations

$$a_{is} a_{is'} = \delta_{ss'} \qquad (s, s' = 2, 3;\ i = f, g).$$

The unit vectors $\boldsymbol{\beta}_2$, $\boldsymbol{\beta}_3$ are defined as

$$\boldsymbol{\beta}_2 = (\boldsymbol{\varkappa}_0 \wedge \boldsymbol{k}) / [1 - (\boldsymbol{\varkappa}_0 \cdot \boldsymbol{k})^2]^{1/2}, \qquad \boldsymbol{\beta}_3 = \boldsymbol{\varkappa}_0 \wedge \boldsymbol{\beta}_2.$$

$\boldsymbol{\kappa}_0 = (\sin\theta \cos\varphi', \sin\theta \sin\varphi', \cos\theta)$ is a unit vector in the direction of the photon momentum vector and $\boldsymbol{k}$ is a unit vector along the z axis.

It is easy to see that the vectors $\boldsymbol{\beta}_2, \boldsymbol{\beta}_3, \boldsymbol{\varkappa}_0$ form a right-handed orthonomal set and satisfy the condition of transversality of the radiation.

x, γ are real parameters which determine the character of the polarisation. In general, (6) describes an elliptical polarisation. But $x = \pi/2, \gamma = 0$ represents a plane polarisation and $x = \pi/4, \gamma = \pi/2$ corresponds to a circular polarisation.

4. Evaluation of the energy of radiation

We note that $\boldsymbol{\varkappa} \cdot \boldsymbol{r} = \varkappa r \sin\theta \sin(\varphi' - \varphi + \pi/2)$, the angle φ is in the coordinate space of the particle whereas φ' is in the momentum space of the photon.

We make use of the result

$$e^{ix\sin\psi} = \sum_{\lambda=-\infty}^{\infty} J_\lambda(x)\, e^{i\lambda\psi}$$

where $J_\lambda(x)$ is the Bessel function of the first kind.

Then if f represents the normalised radial wavefunction including the spin amplitude,

$$\begin{aligned}\boldsymbol{\alpha}_0 \boldsymbol{\beta}_2 &= \frac{1}{2\pi} \int_0^\infty r \, dr \int_0^{2\pi} d\varphi \, [i(m_j - m_j')\varphi - i\boldsymbol{\varkappa} \cdot \boldsymbol{r}] f_j \boldsymbol{\alpha} \boldsymbol{\beta}_2 f_i \\ &= (sD_1 + s'D_2) \int_0^\infty r \, dr \, (I'_{n',m'}(r), I_{n,m+1}(r) + I'_{n',m'+1}(r) I_{n,m}(r)) J_\lambda(ry)\, e^{i\tilde{\lambda}\psi}\end{aligned} \tag{12}$$

where $\psi = \varphi' - \pi/2$, $\tilde{\lambda} = m - m'$.

A similar calculation gives

$$\alpha_0\beta_3 = \mathrm{i}(sD_1 + s'D_2)\int_0^\infty r\,\mathrm{d}r\,[(I'_{n,m}(r)I_{n,m+1}(r) - I'_{n,m+1}(r)I_{n,m}(r))\cos\theta$$
$$+ (I'_{n,m}(r)I_{n,m}(r) - I'_{n,m+1}(r)I_{n,m+1}(r))\sin\theta]J_{\tilde\lambda}(ry)\,\mathrm{e}^{\mathrm{i}\lambda\psi}. \qquad (13)$$

The summation sign has been dropped (with respect to $\tilde\lambda$). In integrating (12) and (13) with respect to φ we made use of the fact that

$$\frac{1}{2\pi}\int_0^{2\pi}\exp[\mathrm{i}(m_j - m'_j - \lambda)\varphi]\,\mathrm{d}\varphi = \delta_{\lambda,m_j-m'_j}$$

which implies that

$$\lambda = m - m' \pm 1, \qquad \lambda = m - m'.$$

Also

$$4D_1^2 = (1 - k_0/K')(1 + k_0/K), \qquad 4D_2^2 = (1 - k_0/K)(1 + k_0/K').$$

Further, we indicate quantities of the final states by means of a prime; thus K'. Those without primes refer to the initial state.

The role of helicity flip can be seen in (12) and (13) in the factor $sD_1 + s'D_2$; the matrix element for helicity flip ($s' = -s$) evidently vanishes in the limit of zero photon energy.

In order to integrate (12) and (13) with respect to r we make use of formula 7.422(2) from Gradshteyn and Ryzhik (1971),

$$\int_0^\infty x^{\lambda+1}\exp(-ax^2)L_n^\sigma(\alpha x^2)L_n^{\lambda-\sigma}(\alpha x^2)J_\lambda(xy)\,\mathrm{d}x$$
$$= (-1)^{n+n'}(2\alpha)^{-\lambda-1}y^\lambda\exp(-y^2/4\alpha)L_n^{\sigma-n'+n}(y^2/4\alpha)L_n^{\lambda-\sigma-n+n'}(y^2/4\alpha).$$

Noting that $\lambda = m_j - m'_j$ we also use the relation

$$L_n^\alpha(z) = (-z)^{-\alpha}L_{n+\alpha}^{-\alpha}(z).$$

After some calculation, substituting (12), (13) into (11) and then (10) we obtain for the nth quantum number

$$\mathrm{d}W = \sum_{i=f,g}\mathrm{d}W_i$$

where

$$\mathrm{d}W_i = \tfrac{1}{4}e^2c(sD_1 + s'D_2)^2\varkappa_-^2\sin\theta\,\mathrm{d}\theta$$
$$\times\Bigg[(I_{n,\nu-1}(\bar z) - I_{n,\nu+1}(\bar z))^2\begin{pmatrix}\cos^2 x\\ \sin^2 x\end{pmatrix}$$
$$+ [(I_{n,\nu-1}(\bar z) + I_{n,\nu+1}(\bar z))\cos\theta + 2I_{n,\nu}(\bar z)\sin\theta]^2\begin{pmatrix}\sin^2 x\\ \cos^2 x\end{pmatrix}$$
$$+ \begin{pmatrix}+1\\ -1\end{pmatrix}\sin 2x\sin\gamma(I_{n,\nu-1}(\bar z) - I_{n,\nu+1}(\bar z))$$
$$\times[(I_{n,\nu-1}(\bar z) + I_{n,\nu+1}(\bar z))\cos\theta + 2I_{n,\nu}(\bar z)\sin\theta]\Bigg] \qquad (14)$$

with $\bar z = (\varkappa^2\sin^2\theta/4\alpha)$. The upper line refers to the component $\mathrm{d}W_f$ and the lower line to dW_g. Further, in obtaining (14) we had summed with respect to all final states n' and in this regard used the result

$$\sum I^2_{n',\nu}(z) = 1$$

(see Sokolov *et al* 1968, Neumann 1953). We also put $n \pm 1 \approx n$ since n is much larger than unity.

Note on (14). From the δ function in (10) we get the equation for the conservation of energy

$$K_{n,m_j} - K_{n',m_j'} - \varkappa = 0.$$

Solving this and making use of (6), we obtain

$$\varkappa^2 - 2\varkappa K + 2\alpha\nu = 0$$

where

$$\nu = n - n' + m - m'.$$

Thus $\varkappa_-$ appearing in (14) is the root of the above equation for which $K \geqslant \varkappa$. The other root corresponding to the case $K < \varkappa$ is rejected on physical grounds. Thus

$$\varkappa_- = K[1 - (1 - 2\alpha\nu/K^2)^{1/2}],$$

i.e.

$$\omega_- = (E/\hbar)[1 - (1 - \mu c^2\hbar\tilde{\omega}\nu/E^2)^{1/2}]. \tag{15}$$

Expanding (15) and making $\hbar \to 0$, we get

$$\omega_{cl} = (\mu c^2/E)\tilde{\omega}\nu$$

which agrees with the relativistic classical value expressing the fact that the frequency of radiation is harmonic to the frequency of rotation of the particle.

From (15), it is obvious that the frequency has real values if

$$1 \geqslant \mu c^2\hbar\tilde{\omega}\nu/E^2,$$

i.e. if

$$\nu \leqslant E^2/e_0\mathscr{H}\hbar c.$$

Hence

$$\nu_{max} = E^2/e_0\mathscr{H}\hbar c.$$

Further, taking note of the selection rule for the magnetic quantum number, i.e. $m - m' = 0, \mp 1$, we see that $\nu_{max} \approx n - n'$. For a given n, $\nu = 0$ when $n = n'$ and attains its maximum when $n' = 0$. Thus

$$\nu_{max} = n = E^2/e_0\mathscr{H}\hbar c. \tag{16}$$

5. Analysis of the polarisation

For an analysis of the polarisation we now introduce the Stokes parameters. They are given by

$$\begin{pmatrix} \mathbf{1} \\ \boldsymbol{P} \end{pmatrix}$$

which has been normalised so as to make

$$|\boldsymbol{P}| \leqslant 1.$$

In the present problem, the components are given by

$$P_1 = \frac{(I_{n,\nu-1}(\bar{z}) - I_{n,\nu+1}(\bar{z})^2 - [(I_{n,\nu-1}(\bar{z}) + I_{n,\nu+1}(\bar{z}))\cos\theta + 2I_{n,\nu}(\bar{z})\sin\theta]^2}{I},$$

$$P_2 = 0, \qquad (17)$$

$$P_3 = \frac{2(I_{n,\nu-1}(\bar{z}) - I_{n,\nu+1}(\bar{z}))[(I_{n,\nu-1}(\bar{z}) + I_{n,\nu+1}(\bar{z}))\cos\theta + 2I_{n,\nu}(\bar{z})\sin\theta]}{I},$$

with

$$I = (I_{n,\nu-1}(\bar{z}) - I_{n,\nu+1}(\bar{z}))^2 + [(I_{n,\nu-1}(\bar{z}) + I_{n,\nu+1}(\bar{z}))\cos\theta + 2I_{n,\nu}(\bar{z})\sin\theta]^2.$$

The component P_1 corresponds to the degree of linearly (plane) polarised radiation. P_2 gives the degree of linearly (plane) polarised component at an angle $\pi/4$ to that of P_1 and P_3 represents a circularly polarised component.

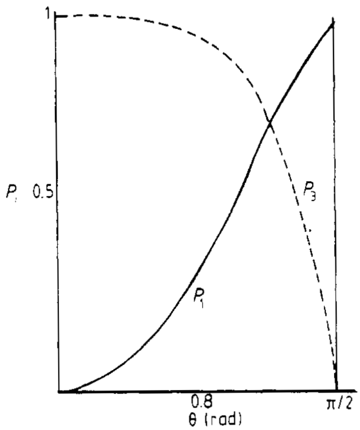

Figure 1. Graph showing degree of polarisation for the case of large quantum number (n). Polar angle θ in radians. P_1 linear polarisation; P_3, circular polarisation.

If $|\boldsymbol{P}| = 1$, we say that the radiation is totally polarised. If $|\boldsymbol{P}| < 1$, it is partially polarised. It is easy to see from (17) that if $\theta = 0$ then $P_1 = 0$ and $P_3 = 1$ which means that radiation along the z axis is totally circularly polarised. On the other hand if $\theta = \pi/2$, $P_1 = 1$, and $P_3 = 0$. In this case the radiation is totally linearly polarised. It is easy to see from (17) that, in general,

$$|\boldsymbol{P}| = (P_1^2 + P_2^2 + P_3^2)^{1/2} = 1.$$

Thus, synchrotron radiation by a Dirac particle is, generally speaking, totally polarised. This is qualitatively in agreement with the classical result (see Kukanov *et al* 1971) and also with the experimental results (see Chrien *et al* 1980, Lea 1978).

Qualitatively, both the classical and the quantum treatments show that the radiation is totally polarised, and that the component $P_2 = 0$. However, we may note the following differences between the two results. The classical expressions contain Bessel functions, whereas the quantum result contains Laguerre polynomials. In the limit as $\hbar \to 0$, however, the Laguerre polynomials go over into the Bessel functions and the classical results are recovered, except for the appearance of an additional term in the expression for the matrix element proportional to $2 \sin\theta\, I_{n,\nu}(z)$ (see (14)). The result of this is that the quantum expression for P_1 is decreased, whereas that of P_3 is increased. In order to explain this situation, we note that the classical expressions are for a spinless particle, whereas the quantum treatment takes account of the spin. We may infer from this that the effect of the spin is to reduce the linearly polarised component, and increase the circularly polarised component. The state of polarisation as a whole, however, remains unchanged.

For a more qualitative picture of the polarisation we shall consider the case for large n for which we use the asymptotic expression for the Laguerre polynomial. We note first of all that

$$n = E^2/e_0\mathscr{H}\hbar c.$$

Taking $E = 20$ GeV, $\mathscr{H} = 10^6$ G, we find that

$$n \approx 10^{16}$$

which is quite a large number. An asymptotic form of the Laguerre polynomial is therefore justified. Thus we have

$$L_n^\alpha(x) = \frac{1}{\sqrt{\pi}} e^{x/2} x^{-\alpha/2-1/4} n^{\alpha/2-1/4} \cos(2\sqrt{nx} - \alpha\pi/2 - \pi/4).$$

Ignoring the normalisation constant, we have

$$I_{n,\nu-1}(z) - I_{n,\nu+1}(z) = [n^{(\nu+1)/2}/\sqrt{\pi}(nx)^{1/4}] \sin(2\sqrt{nx} - \nu\pi/2 - \pi/4)(1/n + 1),$$

$$I_{n,\nu-1}(z) + I_{n,\nu+1}(z) = [n^{(\nu+1)/2}/\sqrt{\pi}(nx)^{1/4}] \sin[2\sqrt{nx} - \nu\pi/2 - \pi/4](1/n - 1),$$

$$I_{n,\nu}(z) = [n^{(\nu+1)/2}/\sqrt{\pi}(nx)^{1/4}] \cos(2\sqrt{nx} - \nu\pi/2 - \pi/4)/\sqrt{n}.$$

Since $|\cos x| \leq 1$, $|\sin x| \leq 1$ for large n, we may reject $(\cos x)/n$, $(\sin x)/n$, in comparison with $\cos x$, $\sin x$. Hence, substituting the above expressions into the values for the P_i, and rejecting terms proportional to $(\cos x)/n$, we find that the Stokes parameters assume the very simple forms

$$P_1 = (\sin^2\theta)/(1+\cos^2\theta), \qquad P_2 = 0, \qquad P_3 = (2\cos\theta)/(1+\cos^2\theta). \tag{18}$$

Again, as can be verified,

$$|\boldsymbol{P}| = (P_1^2 + P_2^2 + P_3^2)^{1/2} = 1,$$

indicating a totally polarised radiation. It is of interest to note that in this limit, the components of the polarisation vector are all independent of the energies, either of the particle, or of the photon.

6. Conclusion

In this article we examined the polarisation properties of synchrotron radiation by a Dirac particle. We first obtained a solution, to show that each component of the wavefunction is associated with either of two possible spin states.

We then obtained an expression for the instantaneous power of radiation. With the help of this and making use of the Stokes parameters, we were able to pick out the various components of the polarisation vector.

We showed that for large quantum number, which corresponds to large energy, radiation is totally polarised. The analysis also indicates that synchrotron radiation is generally accompanied by helicity flip. The formula (14) indicates that although helicity flip occurs, it has no effect whatever on the polarisation of the radiation.

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